Vibration Statistics of Thin Plates with Complex Form

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A new mathematical method of calculation of the probabilistic characteristics of mechanical systems with complex geometry is presented. This method has been demonstrated on the example of random vibrating plates. This method is based on the application of certain special fuctions called *R* functions. In order to demonstrate this method, numerical calculations are presented of probabilistic characteristics for plates with complex geometry which have been clamped on the edge.

Nomenclature

A,B_j,L	= linear differential operators
a_0,a_1	= certain positive constants
C^∞	= infinitely differentiable function space
$c_{lm}(t,\gamma)$	= random functions in linear combination approximating the solution of problem
D_i^k	= multi-indicatory differentiation symbol
D	= cylindrical stiffness of plate
E	= Young's modulus
$f(\bar{x},t,\gamma)$	= measured time-space random field
$\bar{f}(x^1...x^n)$	= R function
$F(X_1...X_n)$	= logic function n variables
h	= plate thickness
K	= operator appearing in structure solution
$K_f(x_1,y_1,t_1,x_2,y_2,t_2)$	= correlation function of random excitation
n	= damping coefficient
P	= probabilistic measure
R^1_+	= positive axis real numbers
R^n	= n-dimensional Euclidean space
$S_2(x)$	= Heaviside generalized function
$T_l(\alpha x), T_m(\beta y)$	= Tschebyshev's multinomials of first type
$V_{kj}(x,y)$	= base function
$\tilde{V}_{kj}(x,y)$	= base function after orthogonalization
u	= solution of considered stochastic boundary-value problem, in particular random translocation of plate
u^*	= normalization of function
t	= time
$\bar{x}$	= coordinate in space R^n, in particular $x^1=x$, $x^2=y$
α,β	= normalizing coefficients
Γ	= set of elementary sentences
γ	= elementary sentence
F	= σ algebra
(Γ,F,P)	= probabilistic space
ρ	= plate material density
$\sigma_u^2(x,y,t)=v_n(x,y,t)$	= variance of plate translation
Ω	= closed limited area in R^n
$\partial\Omega$	= edge of area Ω
ν_0	= Poisson's number
ν	= normal to the edge
τ	= tangent to the edge
$\varphi_i \geq 0$	= inequalities determining elementary areas
$\tilde{\varphi}_0,\tilde{\varphi}_1$	= function appearing in structure solution
$\omega(\bar{x})=0$	= equation of $\partial\Omega$
$\omega_i(\bar{x})=0$	= equations of $\partial\Omega_i$
$\bar{\omega}_{kj}$	= frequency of free vibrations
$\langle\cdot\rangle$	= scalar product in space $L^2(\Omega)$
$[\cdot]$, $\{\cdot\}$	= scalar product in conjugated spaces with $L^2(\Omega)$
$\wedge$	= "and" operation (conjunction)
$\vee$	= "or" operation (alternative)
$\cap$	= multiplicity product
$\cup$	= multiplicity addition
$\bigwedge$	= general quantifier
$\bigvee$	= special quantifier

I. Introduction

CORRELATIONAL and spectral analysis methods are widely applicable within the scope of examining reactions of discrete and continuous dynamic systems subjected to random excitations.[1-14] They examine interrelations between the probabilistic characteristics (average value), the correlative function of random excitation processes conveniently referred to as the "input," and random processes describing the state of the system referred to as the "output." In the case of continuous dynamic systems, the random fields constitute the random processes. It is the aim of this work to present a certain new mathematical method for the determination of probabilistic characteristics of continuous dynamic systems with a complex form subject to nonstationary excitations by random fields. These problems arise in various technical problems, e.g., by examining starting vibrations of air and rocket structures. The demonstration of the methods has been carried out for the case of vibrating plates having a complex form and clamped edge.[14] The method discussed, because of the applications of certain special functions introduced by Rvatschev[15] called the *R* functions, enabled solutions of the problem in the form of closed analytical formulas. Applying the *R* functions, one can obtain the equation of the outline of the edge of the area. This area describes practically free geometry. The characteristic feature of the *R* function is that each of them corresponds to a definite logical function; the arguments of the latter are two discrete values, 1 and 0. This feature enables one to apply the contemporary methods of the algebra of logic to the solution of the boundary problems of mathematical physics in the

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fields of complex form. The logic function, which corresponds to the R function, takes the value 1 when the point under investigation lies within the area or on its edge and the value 0 when it is outside the area. A free area with complex form can be set together by means of the multiplicity operations of addition, multiplication, etc., upon the areas with simple form, the equations of which are known. In the case of the R functions, the multiplicity operations will correspond, then, to the logical operations such as the "or" (alternative) operation and the "and" (conjunctive) operation. Thus the logical functions corresponding to the R functions will possess in themselves certain coded information about the changeover (switchover, commutation) of the sign in the equations describing the edge of the areas with complex form.

II. Formulation of the Problem

The stochastic boundary problems of mathematical physics, connected with linear continuous dynamic systems, can be written in the following operational form[16,17]:

$$Au=f(\bar{x},t,\gamma) \tag{1}$$

$$\bigwedge_{0\le j\le m-1} B_j u \Big|_{\partial\Omega}=0 \tag{2}$$

$$\bigwedge_{1\le k\le l} D_t^k u \Big|_{t=+0}=0 \tag{3}$$

in the area $Q=\Omega x(0,T)$ $(\bar{x}\epsilon\Omega,\ t\epsilon R^1_+,\ 0<t<T)$, where $\Omega\subset R^n$ (n-dimensional Euclidean space), $T<\infty$, and $\partial\Omega$ is the edge of the area Ω. $f(\bar{x},t,\gamma)$ are the exciting measurable random fields induced by probabilistic space (Γ,F,P)[18]; $\gamma\epsilon\Gamma$, Γ is the set of elementary sentences; F is the σ algebra; P is the probabilistic measure in the probabilistic space (Γ,F,P); A, B_j are the linear differential operators in Eq. (1) and the boundary conditions in Eq. (2); D_t^k is the multi-indicatory symbol of the differentiation of the relative time t.

In the case of linear, continuous dynamic systems that vibrate randomly, the operator A appears most often in the shape of

$$Au\equiv a_0\frac{\partial^2 u}{\partial t^2}+a_1\frac{\partial u}{\partial t}+Lu \tag{4}$$

where a_0, a_1 are certain positive constants, and

$$Lu\equiv\sum_{|p|\,|q|\le m}(-1)^{|p|}D^p[a_{pq}(\bar{x})D^q u],\quad a_{pq}(\bar{x})\epsilon C^\infty(\bar{\Omega}) \tag{5}$$

$$D^q=\frac{\partial q}{\partial x^1 q^1 \partial x^2 q^e \ldots \partial x^n q^n},\quad q=q_1+q_2+\ldots+q_n$$

This equation of random vibrations of thin linear plates is a special case of Eq. (1). In the case of variable thickness, it has the shape of

$$\Delta(D\Delta u)-(1-\nu_0)\Big(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-2\cdot\frac{\partial^2 D}{\partial x\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\cdot\frac{\partial^2 u}{\partial x^2}\Big)+\rho h\frac{\partial^2 u}{\partial t^2}+2\rho hn\frac{\partial u}{\partial t}=f(x,y,t,\gamma) \tag{6}$$

and, in the case of constant thickness,

$$D\Delta\Delta u+\rho h\frac{\partial^2 u}{\partial t^2}+2\rho hn\frac{\partial u}{\partial t}=f(x,y,t,\gamma) \tag{7}$$

At the same time,

$$\Delta\Delta=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$$

is the biharmonic operator,

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is Laplace's harmonic operator, and

$$D=\frac{Eh^3}{12(1-\nu_0^2)}$$

is the cylindrical stiffness of the plate.

In this paper, we shall consider the stochastic vibrations of the plates with complex form, which have clamped edges. Their boundary conditions are expressed in the form of

$$u\,|_{\partial\Omega}=0,\qquad \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}=0 \tag{8}$$

where ν is normal to the edge $\partial\Omega$. We shall assume the initial conditions as zero; thus,

$$u(x,y,0)=\frac{\partial}{\partial t}u(x,y,0)=0 \tag{9}$$

III. R Functions and Boundary-Value Problem of Mathematical Physics

The analysis of continuous dynamic systems with complex geometry in the present method requires the description of the complex geometrical form of these systems by means of an equation. It is possible to solve this problem positively by applying the so-called R function.[15] Let us consider the function $S_2(x)$ defined in the following way:

$$S_2(x)\equiv\begin{cases}0 & \text{if}\quad x<0\\ 1 & \text{if}\quad x\ge 0\end{cases} \tag{10}$$

Definition 1 (of the R Function)

The function $y=\tilde{f}(x^1\ldots x^n)$, to which corresponds a Boole's function $F(X_1\ldots X_n)$, such that the relation

$$S_2[\tilde{f}(x^1\ldots x^n)]=F[S_2(x^1)\ldots S_2(x^n)] \tag{11}$$

is fulfilled, is called the R function.

Theorem 1[15]

Let $F(X_1\ldots X_n)$ be a certain Boole's function and $\tilde{f}(x^1\ldots x^n)$ the R function corresponding to it. If the considered closed area consisting of the elementary areas Ω_i, $i=1,2,\ldots,n$, defined by the continuous function $\varphi_i(x^1\ldots x^n)$, is described by the equation

$$F[S_2(\varphi_1),\ S_2(\varphi_2),\ldots,\ S_2(\varphi_n)]=1 \tag{12}$$

then this area may be defined by the inequality

$$\tilde{f}(\varphi_1,\varphi_2\ldots\varphi_n)\ge 0 \tag{13}$$

By making use of Theorem 1, the area Ω with practically free (optional) geometry can be written in the form of an equation. Thus, for instance, if area Ω is an intersection of two areas Ω_1 and Ω_2 defined by the inequalities $\varphi_1\ge 0$, $\varphi_2\ge 0$, $\Omega=\Omega_1\cap\Omega_2$ (Fig. 1), then this area may be defined by the inequality

$$S_2(\varphi_1)\wedge S_2(\varphi_2)=1$$

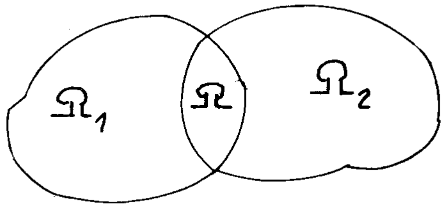

Fig. 1 Area Ω described by the inequality $\varphi_1 \wedge x \varphi_2 > 0$.

According to Theorem 1, the area will be defined by the inequality $\varphi_1 \wedge^* \varphi_2 \geq 0$, where $\wedge^*$ is a symbol of a certain R function. It is important to define the inequalities $\varphi_1, \varphi_2 \ldots \varphi_n$ describing the areas out of which is constructed the sought area Ω. In case the area Ω is two-dimensional, it is convenient to introduce the notion of the map on the plane.

Definition 2

The point pattern of the intersection of surface Ω and the plane $x0y$ is called a map on the plane $x0y$. It has the equation $\bar{f}(x,y)=0$ and intersects with the plane along a certain line $\partial\Omega$, the equation of which has the form $\bar{f}(x,y)=0$. The map corresponds to a certain curve on the plane. The following relations most often are taken as the R function corresponding to various logical operations[15]:

$$\varphi_1 \wedge_\alpha \varphi_2 = [1/(1+2)](\varphi_1 + \varphi_2 - \sqrt{\varphi_1^2 + \varphi_2^2 - 2\alpha\varphi_1\varphi_2}) \tag{14}$$

$$\varphi_1 \vee_\alpha \varphi_2 = [1/(1+2)](\varphi_1 + \varphi_2 + \sqrt{\varphi_1^2 + \varphi_2^2 - 2\alpha\varphi_1\varphi_2}) \tag{15}$$

$$\sim\varphi_1 = -\varphi_1, \quad -1 < \alpha \leq 1 \tag{16}$$

Example 1

A rectangle can be written as the intersection of $\Omega_1 = (a^2 - x^2) \geq 0$, a vertical band, and $\Omega_2 = (b^2 - y^2) \geq 0$ a horizontal band. Thus, $\Omega = \Omega_1 \cap \Omega_2$, and $\Omega = (a^2 - x^2) \wedge_0 (b^2 - y^2)$. Applying the R functions for conjunction ("and" operation), we obtain the equation of the contour of the area Ω in the form

$$\omega(x,y) = a^2 - x^2 + b^2 - y^2 - \sqrt{(a^2 - x^2)^2 + (b^2 - y^2)^2} = 0$$

The equation of the contour should be normalized to the first order, i.e., should fulfill the following conditions:

$$\omega|_{\partial\Omega} = 0 \tag{17}$$

$$\omega \geq 0 \tag{18}$$

where $\bar{x} \epsilon \Omega \cup \partial\bar{\Omega}$, and

$$\frac{\partial\omega}{\partial\nu}\bigg|_{\partial\Omega} = 1 \tag{19}$$

The equation of the contour can be normalized by making use of the formula

$$\omega_1 \equiv \frac{\omega}{\sqrt{\omega^2 + |\mathrm{grad}\ \omega|^2}} = 0 \tag{20}$$

Example 2

The equation of the circle with the radius R and the center $0_1(a,b)$ which is normalized to the first order has the form

$$\omega(x,y) = (1/2R)[R^2 - (x-a)^2 - (y-b)^2] = 0$$

We shall now introduce the notion of the structure of solution.

Definition 3

Let $\tilde{\varphi}_1$ be the element of the linear space R, and $\tilde{\varphi}_0$ the known function belonging to the domain D_A of the operator A in Eq. (1). Let K be the known operator projecting the space $R \rightarrow \Omega x(0,T)$. The expression $u = K(\tilde{\varphi}_1) + \tilde{\varphi}_0$ is called the structure fulfilling the boundary conditions (2) if, with an optional $\tilde{\varphi}_1$, the function u fulfills the boundary conditions (2).

We now shall introduce operators D_K, D_K^τ as the extension of differential operator in relation to the normal ν and the tangent τ on the boundary of the area $\partial\Omega$ to the inside of the area Ω. In the case of two variables, these operators have the form

$$D_K = \left(\frac{\partial\omega}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\omega}{\partial y}\frac{\partial}{\partial y}\right)^K \tag{21}$$

$$D_K^\tau = \left(\frac{\partial\omega}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\omega}{\partial x}\frac{\partial}{\partial y}\right)^K \tag{22}$$

If the equation of the contour must be normalized to higher orders, i.e.,

$$\frac{\partial^2\omega}{\partial\nu^2}\bigg|_{\partial\Omega} = \ldots = \frac{\partial^K\omega}{\partial\nu^K}\bigg|_{\partial\Omega} = 0 \tag{23}$$

then it is possible to emply the following recurrence formula (when normalized):

$$\omega_K = \omega_{K-1} - (1/K!)\omega^K D_K \omega_{K-1} \tag{24}$$

In order to select the structure of the solution in the considered boundary problems, we shall develop the solution in a Taylor's series around the contour described by $\omega(\bar{x})$.

Theorem 2[15]

If the function $\omega(\bar{x})$ is normalized to the order of $(n+1)$, then

$$u(\bar{x},t,\gamma) = u^*(\bar{x},t,\gamma) + \sum_{k=1}^{n} \frac{1}{k!} u_k^*(\bar{x},t,\gamma)\omega^k(\bar{x}) + O[\rho^{h+1}(\bar{x})] \tag{25}$$

where

$$u^*(\bar{x},t,\gamma) = u[\bar{x} - \omega(\bar{x})\nabla\omega(\bar{x}),t,\gamma] \tag{26}$$

is called the normalization of the function $u(\bar{x},t,\gamma)$ in relation to the function $\omega(\bar{x})$.

By making use of the preceding expansion and of the boundary conditions, it is easy to deduce the structure of the solutions for various boundary problems of mathematical physics. We shall show it in the case of a plate clamped on the whole edge. The boundary conditions for such a plate have the form

$$u|_{\partial\Omega} = 0 = \frac{\partial u}{\partial\nu}\bigg|_{\partial\Omega} = 0$$

If u_0^* and u_1^* are normalizations in relation to the contour normalized to first order, then

$$u|_{\partial\Omega} = u_0|_{\partial\Omega}, \quad \frac{\partial u}{\partial\nu}\bigg|_{\partial\Omega} = u_1|_{\partial\Omega}$$

Since u has to fulfill the boundary conditions $u_0 = 0$, $u_1 = 0$, and hence $u_0^* = 0$ and $u_1^* = 0$, so the structure of the solution is expressed by the formula

$$u(x,y,t,\gamma) = \omega^2(x,y)\,½!\,u_2^*(x,y,t,\gamma) = \omega^2(x,y)\psi(x,y,t,\gamma) \tag{27}$$

The unknown fucntions $\psi(x,y,t,\gamma)$ can be presented in the form

$$\psi(x,y,t,\gamma) = \sum_{l+m=0}^{n} c_{lm}(t,\gamma)\psi_{lm}(x,y) \tag{28}$$

where

$$\psi_{lm}(x,y) = T_l(\alpha x)T_m(\beta y)$$

T_l, T_m are Tschebyschev's multinomials of the first type; and α, β are normalizing coefficients. They depend on the localization of the origin of coordinates xy and on the dimension of area Ω. The unknown random functions $c_{lm}(t,\gamma)$ are sought by means of Galerkin-Ritz's method.

IV. Method of Solution

Applying Galerkin-Ritz's method, we shall present the solution of the boundary problem (7-9) in the form

$$u_n(x,y,t,\gamma) = \sum_{l+m=0}^{n} c_{lm}(t,\gamma)V_{lm}(x,y) \tag{29}$$

The base functions $V_{lm}(x,y)$ in the case of a clamped plate have the form

$$V_{lm}(x,y) = \omega^2(x,y)\psi_{lm}(x,y) \tag{30}$$

where

$$\sum_{l+m=0}^{n} = \sum_{l=0}^{n}\sum_{m=0}^{n-l}$$

and where $\omega(x,y)$ is the equation of the contour of the plate, normalized to the first order, written by means of the R function; and $\psi_{lm}(x,y) = T_l(\alpha x)T_m(\beta y)$.

The random functions $c_{lm}(t,\gamma)$ for a plate of constant thickness are calculated on the basis of the following system of ordinary differential equations:

$$\sum_{l+m=0}^{n}\left\{c_{lm}\iint_\Omega D\left[\Delta V_{lm}\Delta V_{kj} - (1-\nu_0)\left(\frac{\partial^2 V_{lm}}{\partial x^2}\frac{\partial^2 V_{kj}}{\partial y^2} + \frac{\partial^2 V_{kj}}{\partial x^2}\frac{\partial^2 V_{lm}}{\partial y^2} - 2\frac{\partial^2 V_{lm}}{\partial x\partial y}\frac{\partial^2 V_{kj}}{\partial x\partial y}\right)\right]\mathrm{d}x\mathrm{d}y + \left[2\rho hn\frac{\mathrm{d}c_{lm}}{\mathrm{d}t} + \rho h\frac{\mathrm{d}^2 c_{lm}}{\mathrm{d}t^2}\right]\iint_\Omega V_{lm}V_{kj}\mathrm{d}x\mathrm{d}y\right\} = \iint_\Omega f(x,y,t,\gamma)V_{kj}\mathrm{d}x\mathrm{d}y \tag{31}$$

This system, in the case of a plate with clamped edge, resolves itself into the form

$$\sum_{l+m=0}^{n}\left\{c_{lm}\iint_\Omega D\Delta V_{lm}\Delta V_{kj}\mathrm{d}x\mathrm{d}y + \left[2\rho hn\frac{\mathrm{d}c_{lm}}{\mathrm{d}t} + \rho h\frac{\mathrm{d}^2 c_{lm}}{\mathrm{d}t^2}\right]\times\iint_\Omega V_{lm}V_{kj}\mathrm{d}x\mathrm{d}y\right\} = \iint_\Omega f(x,y,t,\delta)V_{kj}\mathrm{d}x\mathrm{d}y \tag{32}$$

The eigenvalues of the considered boundary problem can be calculated on the basis of the following symmetrical matrix determinant:

$$\det\|[V_{lm},V_{kj}] - \lambda\langle V_{lm},V_{kj}\rangle\| = 0 \tag{33}$$

where

$$[V_{lm},V_{kj}] = \iint_\Omega \frac{D}{\rho h}\left[\Delta V_{lm}\Delta V_{kj} - (1-\nu_0)\left(\frac{\partial^2 V_{lm}}{\partial x^2}\frac{\partial^2 V_{kj}}{\partial y^2} + \frac{\partial^2 V_{kj}}{\partial x^2}\frac{\partial^2 V_{kj}}{\partial y^2} - 2\frac{\partial^2 V_{lm}}{\partial x\partial y}\frac{\partial^2 V_{kj}}{\partial x\partial y}\right)\right]\mathrm{d}x\mathrm{d}y \tag{34}$$

$$\langle V_{lm},V_{kj}\rangle = \iint_\Omega V_{lm}V_{kj}\mathrm{d}x\mathrm{d}y \tag{35}$$

and, for plates having clamped edge,

$$[V_{lm},V_{kj}] = \iint_\Omega \frac{D}{\rho h}\Delta V_{lm}\Delta V_{kj}\mathrm{d}x\mathrm{d}y \tag{36}$$

Knowing the eigenvalues λ_{lm}, it is possible to orthogonalize the base functions $V_{lm}(x,y)$ in accordance with scalar product:

$$\{V_{lm},V_{kj}\} \equiv [V_{lm},V_{kj}] - \lambda_{lm}\langle V_{lm},V_{kj}\rangle \tag{37}$$

and to normalize in accordance with the norm $|V_{lm}| = \sqrt{\langle V_{lm},V_{lm}\rangle}$. In the case of multiple eigenvalues λ_{lm}, the secondary orthogonalization of the function V_{lm} according to scalar product $\langle\cdot\rangle$ should be carried out.

Obtained in this way, new base functions $\tilde{V}_{kj}(x,y)$ have the property that the system of ordinary differential equations (31) and (32), has the canonic form

$$\ddot{c}_{kj} + 2n\dot{c}_{kj} + \tilde{\omega}_{kj}^2 c_{kj} = f_{kj}(t,\gamma) \tag{38}$$

where

$$\ddot{c}_{kj} = \frac{\mathrm{d}^2 c_{kj}}{\mathrm{d}t^2},\qquad \dot{c}_{kj} = \frac{\mathrm{d}c_{kj}}{\mathrm{d}t}$$

$$\tilde{\omega}_{kj}^2 = \lambda_{kj} = [\tilde{V}_{kj},\tilde{V}_{kj}]$$

$$f_{kj}(t,\gamma) = \iint_\Omega \frac{1}{\rho h}f(x,y,t,\gamma)\tilde{V}_{kj}(x,y)\mathrm{d}x\mathrm{d}y$$

The functions $\tilde{V}_{kj}(x,y)$ have the form $V_{kj}(x,y)$ after orthogonalization.

In the case of weak damping when $\tilde{\omega}_{kj} \gg n$, only this sort of damping will be analyzed; the final solution is defined by the formula

$$u_n(x,y,t,\gamma) = \sum_{l+m=0}^{n}\int_0^t\iint_\Omega e^{-n(t-\tau)}\frac{\sin p_{lm}(t-\tau)}{\rho h p_{lm}}f(\xi,\zeta,\tau,\gamma)\times\tilde{V}_{lm}(\xi,\zeta)\mathrm{d}\xi\mathrm{d}\zeta\mathrm{d}\tau\tilde{V}_{lm}(x,y) \tag{39}$$

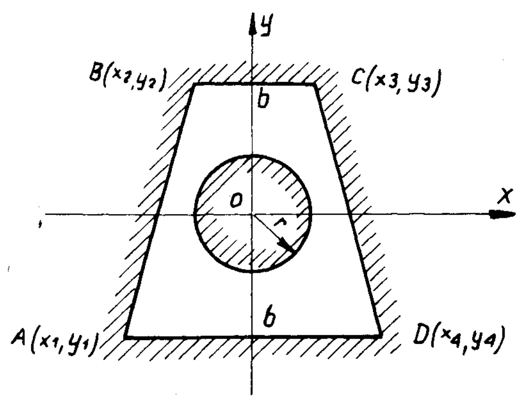

Fig. 2 Trapezoidal plate with circular hole cut out in the middle, with clamped edge.

where $p_{lm} = \sqrt{\bar{\omega}_{lm}^2 - n^2}$. Green's function or the impulse function of transition for the preceding solution has the form

$$G(x,\xi,y,\zeta,t,\tau) = \sum_{l+m=0}^{n} \frac{1}{\rho h} e^{-n(t-\tau)} \frac{\sin p_{lm}(t-\tau)}{p_{lm}}$$

$$\times \tilde{V}_{lm}(\xi,\zeta)\tilde{V}_{lm}(x,y) \tag{40}$$

The variance of translation of a plate is expressed by the formula

$$\sigma_u^2(x,y,t) = \nu_n(x,y,t)$$

$$= \sum_{l+m=0}^{n} \sum_{k+j=0}^{n} \int_0^t \int_0^t \iint_\Omega \iint_\Omega K_f(\xi_1,\zeta_1,\tau_1,\xi_2,\zeta_2,\tau_2)$$

$$\times \tilde{V}_{lm}(\xi_1,\zeta_1)\tilde{V}_{kj}(\xi_2,\zeta_2) G_{lm}(t-\tau_1) G_{kj}(t-\tau_2)$$

$$\times d\xi_1 d\zeta_1 d\tau_1 d\xi_2 d\zeta_2 d\tau_2 \tilde{V}_{lm}(x,y)\tilde{V}_{kj}(x,y) \tag{41}$$

where

$$G_{lm,kj}(t-\tau) = \frac{e^{-n(t-\tau)}}{\rho h p_{lm,kj}} \sin p_{lm,kj}(t-\tau) \tag{42}$$

The correlation functions of the nonstationary random load are expressed by means of the formulas

$$K_f(t_1,t_2) = \sum_{q=1}^{N} \sigma_q^2 \exp[-\alpha_q(t_1^2 + t_2^2)] \tag{43a}$$

and

$$K_f(x_1,y_1,t_1,x_2,y_2,t_2) = \sum_{q=1}^{N} \sigma_q^2 \cos\{\beta_q(t_1^2 - t_2^2)$$

$$-\lambda_q[(x_1 - x_2) + (y_1 - y_2)]\} \tag{43b}$$

Substituting the correlation function expressed by the formulas (43a) and (43b) into the relation (41), we obtain, after enumerating the integrals appearing in it, complicated formulas on the basis of which it is possible to calculate the variance of transation of the plates.

V. Numerical Examples

Numerical calculations were carried out for a trapezoidal plate with a circular hole cut out in the middle of it (as is shown in Fig. 2) having clamped edges. The equation of the contour of the plate written by means of the R functions has the form

$$\omega(x,y) = \varphi_1 \wedge_0 \varphi_2 \wedge_0 \varphi_3 \wedge_0 \varphi_4 \tag{44}$$

where

$$\varphi_1 = \frac{-x(y_2 - y_1) + y(x_2 - x_1) - y_1 x_2 + y_2 x_1}{\sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}} \geq 0$$

$$\varphi_2 = \frac{-x(y_4 - y_3) + y(x_4 - x_3) - y_3 x_4 + x_3 y_4}{\sqrt{(y_3 - y_4)^2 + (x_3 - x_4)^2}} \geq 0$$

$$\varphi_3 = \frac{1}{2b}(b^2 - y^2) \geq 0, \quad \varphi_4 = -\frac{1}{2r}(r^2 - x^2 - y^2) \geq 0$$

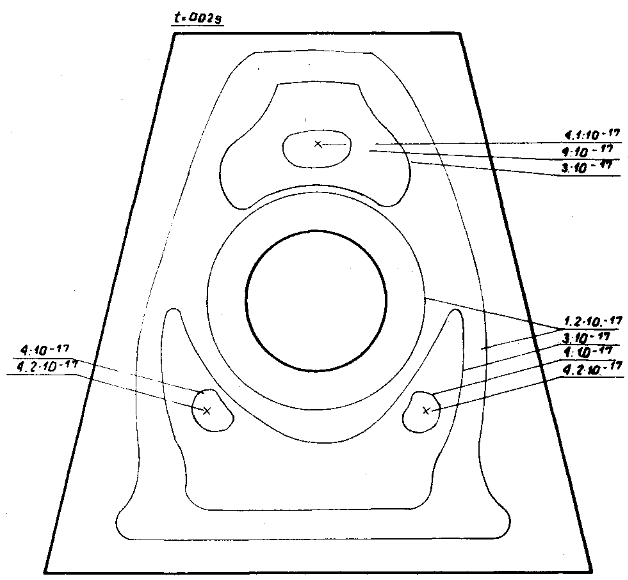

Fig. 4 Contour lines of the variance of the plate translation for the correlation function, Eq. (43a), for time $t_2 = 0.02$ s.

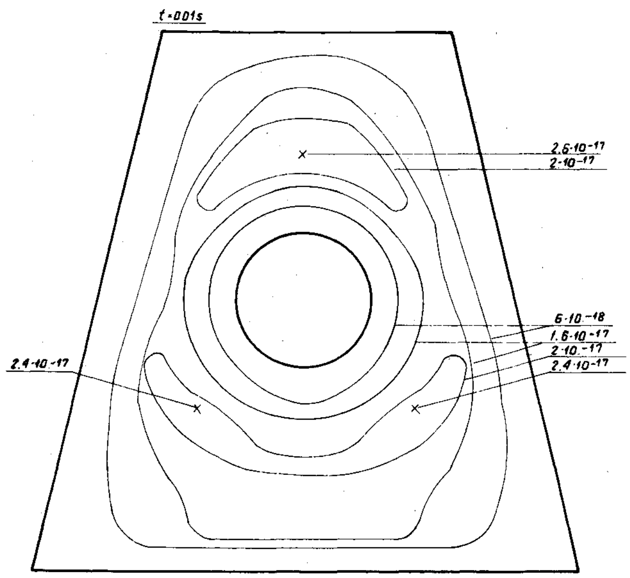

Fig. 3 Contour lines of the variance of the plate translation for the correlation function, Eq. (43a), for time $t_1 = 0.01$ s.

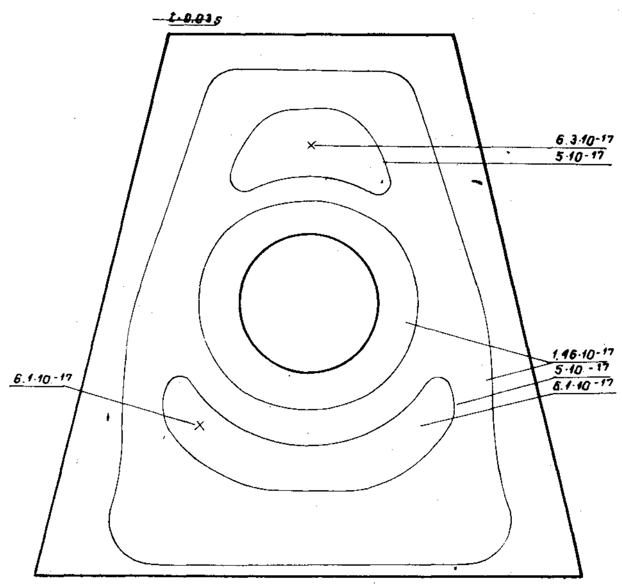

Fig. 5 Contour lines of the variance of the plate translation for the correlation function, Eq. (43a), for time $t_3 = 0.03$ s.

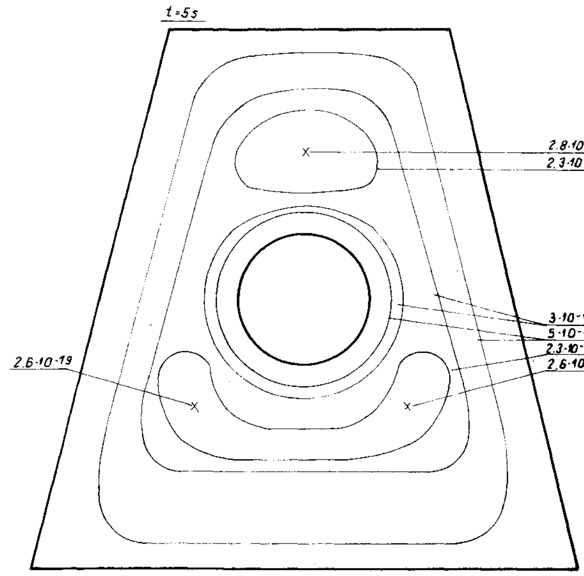

Fig. 6 Contour lines of the variance of the plate translation for the correlation function, Eq. (43a), for time $t_4 = 5$ s.

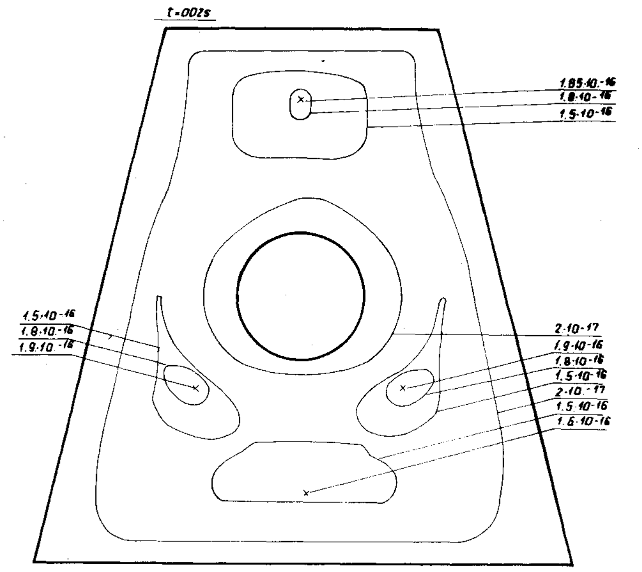

Fig. 8 Contour lines of the variance of the plate translation for the correlation function, Eq. (43b), for time $t_2 = 0.02$ s.

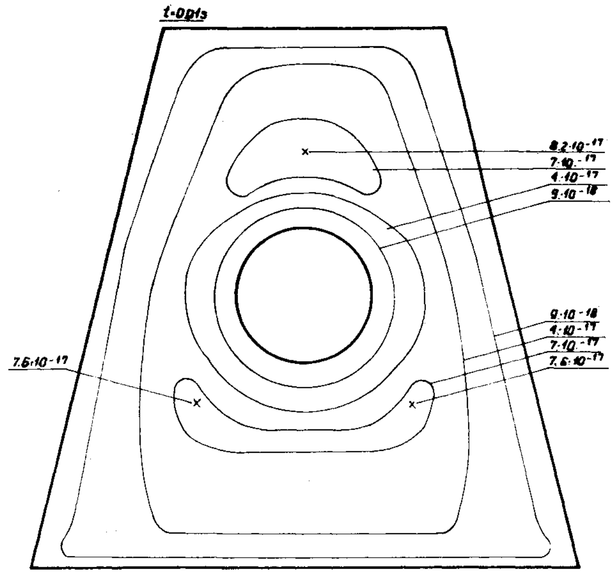

Fig. 7 Contour lines of the variance of the plate translation for the correlation function, Eq. (43b), for time $t_1 = 0.01$ s.

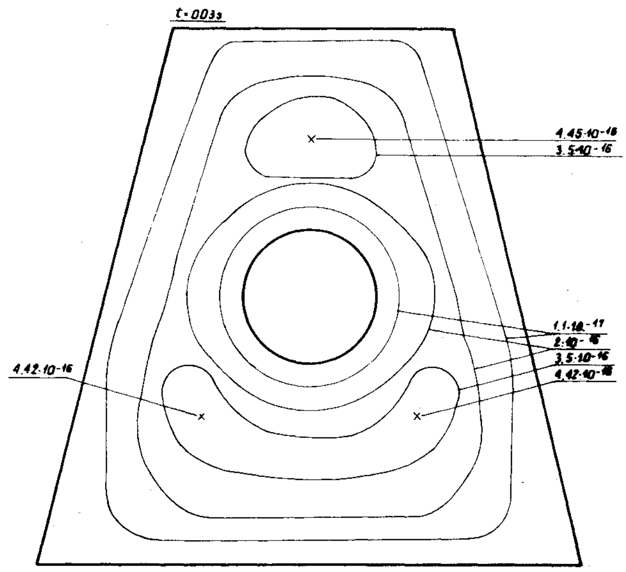

Fig. 9 Contour lines of the variance of the plate translation for the correlation function, Eq. (43b), for time $t_3 = 0.03$ s.

and $\wedge_0$ is the operation of the R function expressed by the formula

$$\varphi_1 \wedge_0 \varphi_2 = \varphi_1 + \varphi_2 - \sqrt{\varphi_1^2 + \varphi_2^2}$$

The coordinates of the corners of the plate have been defined as

$$(x_1, y_1) = A(-1;\ -1)$$

$$(x_2, y_2) = B(-0.5;\ 1)$$

$$(x_3, y_3) = C(0.5;\ 1)$$

$$(x_4, y_4) = D(1;\ -1)$$

The following physical and geometrical parameters have been assumed:

$E = 2 \times 10^{11}\,\mathrm{N/m^2}$, $h = 0.02\,\mathrm{m}$, $\rho = 7800\,\mathrm{kg/m^3}$, $n = 5\,\mathrm{1/s}$

$$\nu_0 = 0.3, \quad b = 1\,\mathrm{m}, \quad r = 0.25\,\mathrm{m}$$

The parameters in the correlation functions (43a) and (43b) have been defined as

$N = 1$

$\sigma_1 = 1$ for Eq. (43a); $\sigma_1 = 2$ for Eq. (43b)

$\beta_1 = 1\,\mathrm{s^{-2}}$; $\alpha_1 = 0.1\,\mathrm{s^{-2}}$; $\lambda_1 = 0.21\,\mathrm{1/m}$

The algorithm of the calculations on a digital computer can be divided into three stages: 1) the numerical calculation of double integrals appearing in Galerkin-Ritz's system of ordinary differential equations[19]; 2) the determination of the frequency of free vibrations and the orthogonalization of the base function of the solution[20]; and 3) the calculation of the variances according to the deduced analytic formulas. The numerical calculations have been carried out mainly in order to investigate the space distribution concerning the variances of the translation of the plate in order to localize its maximal value. Whether or not this localization is variable in time also has been investigated. Further investigations of the variation of variances in time have been omitted as less interesting. The reason for this is that such an analysis would not give much new; it would reduce the problem to the investigation of the

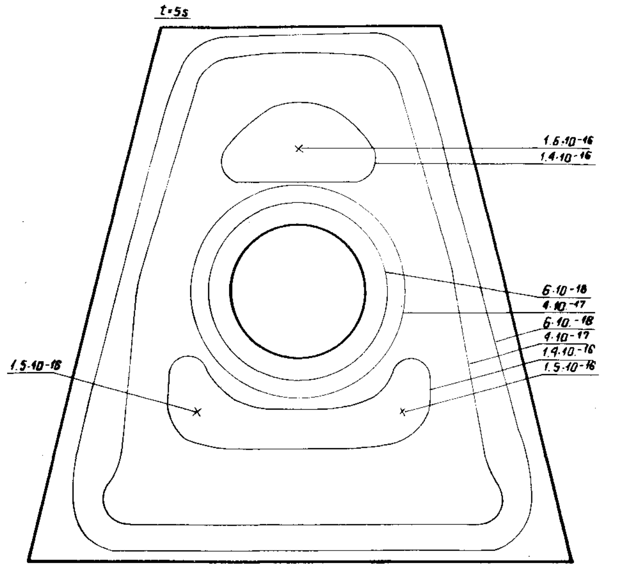

Fig. 10 Contour lines of the variance of the plate translation for the correlation function, Eq. (43b), for time $t_4 = 5$ s.

reactions of a harmonic oscillator upon excitation with nonstationary random processes. The investigations of this type already have been carried out in Refs. 6, 12, and 13.

The variance of the translation of the plate has been shown in the shape of contour lines for the times $t_1 = 0.01$ s, $t_2 = 0.2$ s, $t_3 = 0.03$ s, and $t_4 = 5$ s as in Figs. 3-6 for the random load with the correlation function, Eq. (43a) , and in Figs. 7-10 for the random load with the correlation function, Eq. (43b). The maximum values of the variance in the case of both excitations are located in the middle upper part of the plate and are distributed symmetrically in relation to the vertical axis of coordinate system in the lower part of the plate. This variance is the starting point for further studies concerning the problems of reliability of working plates and other structural elements exposed to random loads.

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